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Color Confinement in Perturbation Theory from a Topological Model

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Abstract

Color confinement by the mechanism of Kugo and Ojima can treat confinement of any quantized color carrying fields including dynamical quarks. However, the non-perturbative condition for this confinement has been known to be satisfied only in the pure-gauge model (PGM), which is a topological model without physical degrees of freedom. Here we analyze the Yang-Mills theory by adding physical degrees of freedom as perturbation to PGM. We find that quarks and gluons are indeed confined in this perturbation theory.

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1 Introduction

The understanding of the color confinement mechanism is one of the most fascinating problems in QCD. In this paper we shall study the color confinement mechanism proposed by Kugo and Ojima (KO) [1]. It is based on the BRST quantized Yang-Mills theory (coupled to the quark field Ψ) described by the Lagrangian,

$$\mathcal{L} = \frac{1}{2g_{\mathrm{YM}}^2}\mathrm{tr}F_{\mu\nu}^2(A) + \overline{\Psi}(i\not{\mathcal{D}}(A) - m)\Psi - i\delta_{\mathrm{B}}G(A_\mu, c, \overline{c}, B), \tag{1.1}$$

with $F_{\mu\nu}(A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $\mathcal{D}_\mu(A) \equiv \partial_\mu + A_\mu$.* In Eq. (1.1), δ_{B} is the BRST transformation, and $G(A_\mu, c, \overline{c}, B)$ specifies the gauge fixing. In this paper we adopt as G either $G_\alpha = \mathrm{tr}\left[\overline{c}\left(\partial^\mu A_\mu - \alpha B\right)\right]$ corresponding to the Feynman type gauge or

$$G_{\mathrm{OSp}} = \frac{2}{\lambda}\delta_{\mathrm{A}}\Big\{\mathrm{tr}\left(A_\mu^2 + 2ic\overline{c}\right)\Big\}, \tag{1.2}$$

of the OSp(4/2) symmetric gauge [2] (λ is the gauge parameter and δ_{A} is the anti-BRST transformation [3] defined in the Appendix).

The confinement by the KO mechanism is in the sense that there are no *physical color-carrying* asymptotic fields. In contrast to the Wilson loop criterion [4], the KO mechanism treats confinement of *quantized* quarks and gluons. According to Kugo and Ojima [1], a sufficient condition for this confinement to be realized is that the BRST-exact conserved color current of the system (1.1),

$$N_\mu = -i\delta_{\mathrm{B}}\Big(D_\mu\overline{c}\Big) = -i\delta_{\mathrm{B}}\Big(\partial_\mu\overline{c} + [A_\mu, \overline{c}]\Big), \tag{1.3}$$

contains no (Nambu-Goldstone-like) massless one-particle mode. If this condition is satisfied, the color charge Q^a is written in a well-defined manner as $Q^a = \{Q_{\mathrm{B}}, \int d^3x\,(D_0\overline{c})^a\}$. The BRST-exact expression $Q^a = \{Q_{\mathrm{B}}, *\}$ ensures that any color non-singlet asymptotic states are necessarily BRST unphysical and hence unobservable.

Differently to the Wilson loop criterion, the KO mechanism seems to have little relationship to the observable QCD dynamics creating mesons and baryons. Due to this property, however, the KO confinement mechanism may be studied rather simply without being bothered by complicated observable dynamics. An extreme example of the realization of this expectation

* We restrict the gauge group to SU(N). The field variables $\phi = A_\mu, c, \overline{c}, B$ are Lie algebra valued and are expressed as $\phi = \sum_{a=1}^{N^2-1}\phi^a t^a$ in terms of Hermitian fields ϕ^a and the (anti-Hermitian) basis t^a with the normalization $\mathrm{tr}(t^a t^b) = -(1/2)\delta^{ab}$.

is the pure-gauge model (PGM) [5, 6]. This is a toy model for the KO confinement mechanism. The PGM is obtained from the Yang-Mills system (1.1) by restricting the gauge field A_μ to the pure-gauge configuration, $A_\mu = g^\dagger \partial_\mu g$, and therefore is described by the "topological" Lagrangian:

$$\mathcal{L}_{\text{PGM}} = -i\delta_{\text{B}} G(g^\dagger \partial_\mu g, c, \overline{c}, B). \tag{1.4}$$

Although the PGM has only unphysical gauge-modes, the KO condition that N_μ is free from the massless mode is still a non-trivial problem of dynamics. If we adopt the OSp(4/2) symmetric gauge with G_{OSp} (1.2), the PGM in four dimensions becomes equivalent to the chiral model in two dimensions owing to the Parisi-Sourlas mechanism [7]. This equivalence and the fact that the chiral model in two dimensions is realized in the disordered phase with a mass gap [8] implies that the KO condition is actually satisfied in the PGM with the OSp(4/2) symmetric gauge [6]. This property is expected to be shared with the ordinary Feynman type gauge [5]. A lesson we learn from PGM is that, among the degrees of freedom in A_μ, the large fluctuation of the gauge-mode $g(x)$, namely the mode in the direction of the gauge transformation, is important for the KO confinement mechanism (cf. Ref. [9]).

The purpose of this paper is to present a method to perturbatively introduce physical degrees of freedom into the PGM and study the KO confinement mechanism in the real Yang-Mills theory. Roughly speaking what we would like to do is to express the gauge field as $A_\mu = g^\dagger \partial_\mu g +$ (physical-modes) and treat the physical-modes as perturbation to PGM. The parameter of our perturbation expansion is simply the gauge coupling constant g_{YM} (note that the PGM is obtained from the Yang-Mills system (1.1) in the vanishing coupling constant limit $g_{\text{YM}} \to 0$). We see that, in this perturbation method, quark and gluon fields are confined by the KO mechanism due to the large fluctuation of the gauge-mode $g(x)$. A similar attempt to adding physical modes to the PGM was made by Izawa [10] using the BF formulation.

The organization of the rest of this paper is as follows. In Sec. 2, we present the general framework of our perturbation expansion from PGM, and in Sec. 3, we apply it to the study of the KO confinement mechanism. In Sec. 4, we generalize our formulation to the finite temperature case. The final section (Sec. 5) is devoted to discussion on the remaining problems. In the Appendix, we summarize the (anti-)BRST transformations.

2 Perturbation expansion around the PGM

For carrying out the perturbation expansion around the PGM, we shall first modify the Yang-Mills system (1.1) by introducing the new auxiliary fields. That is, we rewrite the system in such a way that the gauge field is expressed as a sum of the pure-gauge mode $g^\dagger \partial_\mu g$ and the other (physical) modes. Consider the partition function

$$Z[J] = \int \mathcal{D}A_\mu \mathcal{D}c \mathcal{D}\overline{c} \mathcal{D}B \mathcal{D}\Psi \mathcal{D}\overline{\Psi} \exp\left\{ i \int dx \Big(\mathcal{L} + J \cdot \Phi \Big) \right\}, \tag{2.1}$$

where $\mathcal{L}$ is given by (1.1) and $J \cdot \Phi \equiv \mathrm{tr}\left(J^\mu A_\mu + J_c c + J_{\overline{c}} \overline{c} + J_B B \right) + \overline{j}\Psi + j\overline{\Psi}$ is the source term. Our strategy is to carry out the Faddeev-Popov (FP) trick [11] again to (2.1). We define the FP determinant $\Delta[A]$ as usual by

$$1 = \Delta[A] \int \mathcal{D}g \prod_x \delta\left(\partial^\mu A_\mu^{g^{-1}}(x) \right), \tag{2.2}$$

where $g(x)$ is an element of the gauge group $\mathrm{SU}(N)$ and $A_\mu^g \equiv g^\dagger \left(\partial_\mu + A_\mu \right) g$. We have adopted the Landau gauge for (2.2). We multiply (2.2) to the partition function (2.1) and express the fields A_μ and Ψ as the gauge transform of the new fields a_μ and ψ:

$$\begin{aligned} A_\mu &= a_\mu^g \equiv g^\dagger \partial_\mu g + g^\dagger a_\mu g, \\ \Psi &= \psi^g \equiv g^\dagger \psi. \end{aligned} \tag{2.3}$$

Using the gauge invariance of the FP determinant $\Delta[A] = \Delta[A^g]$ and expressing the product $\Delta[a] \prod \delta\left(\partial^\mu a_\mu(x) \right)$ in terms of the integration over the new (anti-)ghosts γ, $\overline{\gamma}$ and the multiplier field β,

$$\begin{aligned} \Delta[a] \prod_x \delta\left(\partial^\mu a_\mu(x) \right) &= \int \mathcal{D}\gamma \mathcal{D}\overline{\gamma} \mathcal{D}\beta \exp\left\{ i \int dx \left(\beta^a \partial^\mu a_\mu^a + i \overline{\gamma}^a \partial^\mu D_\mu(a) \gamma^a \right) \right\} \\ &= \int \mathcal{D}\gamma \mathcal{D}\overline{\gamma} \mathcal{D}\beta \exp\left\{ i \int dx \left(-i \tilde{\delta}_{\mathrm{B}} H\left(a_\mu, \gamma, \overline{\gamma}, \beta \right) \right) \right\}, \end{aligned} \tag{2.4}$$

the partition function (2.1) is rewritten as

$$\begin{aligned} Z[J] &= \int \mathcal{D}g \mathcal{D}c \mathcal{D}\overline{c} \mathcal{D}B \int \mathcal{D}a_\mu \mathcal{D}\gamma \mathcal{D}\overline{\gamma} \mathcal{D}\beta \mathcal{D}\overline{\psi} \mathcal{D}\psi \\ &\times \exp\left\{ i \int dx \left[\frac{1}{2g_{\mathrm{YM}}^2} \mathrm{tr} F_{\mu\nu}^2(a) + \overline{\psi} \left(i \not{D}(a) - m \right) \psi - i \tilde{\delta}_{\mathrm{B}} H\left(a_\mu, \gamma, \overline{\gamma}, \beta \right) \right. \right. \\ &\qquad \left. \left. - i \delta_{\mathrm{B}} G\left(g^\dagger \left(a_\mu + \partial_\mu \right) g, c, \overline{c}, B \right) + J \cdot \Phi \right] \right\}. \end{aligned} \tag{2.5}$$

In Eqs. (2.4) and (2.5), we have $H = \mathrm{tr}\,(\overline{\gamma}\,\partial^\mu a_\mu)$, and the new BRST transformation $\widetilde{\delta}_{\mathrm{B}}$ is given in the Appendix. A_μ and Ψ in the source term $J \cdot \Phi$ should be understood to be expressed in terms of g, a_μ and ψ as given in Eq. (2.3).

Eq. (2.5) combined with Eq. (2.3) is the basic formula for our perturbation theory. The original gauge field A_μ is expressed in terms of the new gauge field a_μ in Landau gauge and the gauge fluctuation g around a_μ. The path-integrations in (2.5) consist of those over the PGM fields $(g, c, \overline{c}, B)$ and those over the new fields $(a_\mu, \gamma, \overline{\gamma}, \beta, \psi)$. Corresponding to two kinds of the BRST transformations, δ_{B} and $\widetilde{\delta}_{\mathrm{B}}$, the system (2.5) has $\mathrm{SU}(N)_{\mathrm{R}}$ and $\mathrm{SU}(N)_{\mathrm{L}}$ global symmetries; $g(x) \to h_{\mathrm{L}}^\dagger g(x) h_{\mathrm{R}}$. The original fields $(A_\mu, c, \overline{c}, B, \Psi)$ and the new ones $(a_\mu, \gamma, \overline{\gamma}, \beta, \psi)$ are transformed by $\mathrm{SU}(N)_{\mathrm{R}}$ and $\mathrm{SU}(N)_{\mathrm{L}}$, respectively. The color rotation of the original system (1.1) is $\mathrm{SU}(N)_{\mathrm{R}}$.

Our formula (2.5) has no advantage over the original Z (2.1) for calculating gauge invariant quantities. However, in the study of the KO mechanism, we are interested in the gauge-variant quantities such as the Green's function $\langle \mathrm{T} N_\mu A_\nu \rangle$ [1], and our formula will prove very useful as we shall see below.

Now, we shall explain our perturbation expansion around the PGM on the basis of the formulas (2.3) and (2.5). In the following we are interested in the Green's functions of the original fields $(A_\mu = a_\mu^g, c, \overline{c}, B, \Psi = \psi^g)$ obtained by differentiating $Z[J]$ (2.5) with respect to J. Our expansion around the PGM is to carry out the integration over the auxiliary fields $(a_\mu, \gamma, \overline{\gamma}, \beta, \psi)$ in (2.5) using perturbation expansion in powers of the coupling constant g_{YM}^2. The remaining integration in (2.5) over the PGM fields $(g, c, \overline{c}, B)$ should be treated non-perturbatively.

Let us consider this expansion more concretely. First, it should be noticed that in the exponent of (2.5) the interactions between the PGM fields $(g, c, \overline{c}, B)$ and the auxiliary ones $(a_\mu, \gamma, \overline{\gamma}, \beta, \psi)$ come from 1) the gauge fixing term $-i\delta_{\mathrm{B}} G$, 2) the $g^\dagger a_\mu g$ term in $A_\mu = a_\mu^g$ of the source term $J^\mu A_\mu$, and 3) $\Psi = g^\dagger \psi$ in the source term $\bar{j}\Psi + j\overline{\Psi}$. Especially the first contribution 1) is as follows. Taking the OSp(4/2) symmetric gauge, we have

$$\begin{aligned} -i\delta_{\mathrm{B}} G_{\mathrm{OSp}}\Big(g^\dagger\left(a_\mu + \partial_\mu\right) g, c, \overline{c}, B\Big) &= \frac{2i}{\lambda}\delta_{\mathrm{A}}\delta_{\mathrm{B}}\ \mathrm{tr}\Big[(g^\dagger a_\mu g + g^\dagger \partial_\mu g)^2 + 2ic\overline{c}\Big] \\ &= -i\delta_{\mathrm{B}} G_{\mathrm{OSp}}\Big(g^\dagger \partial_\mu g, c, \overline{c}, B\Big) - \frac{4i}{\lambda}\mathrm{tr}\Big[a_\mu\,\delta_{\mathrm{A}}\delta_{\mathrm{B}}(g\partial^\mu g^\dagger)\Big], \end{aligned} \tag{2.6}$$

where use has been made of the fact that a_μ is invariant under both δ_{A} and δ_{B}. In the last expression of (2.6) the first term is nothing but the PGM Lagrangian (1.4), and the second

term is the interaction between the PGM fields and the auxiliary one a_μ. The separation into the PGM Lagrangian and the interaction of the form tr $\left[a_\mu \delta_{\rm A}\delta_{\rm B}(g\partial^\mu g^\dagger)\right]$ is also the case if we take the ordinary Feynman type gauge.

Therefore, in the case $j=\bar{j}=0$ (i.e., no external quarks), the generating functional (2.5) is rewritten as follows:

$$\begin{aligned} Z[J] &= \int \mathcal{D}g\mathcal{D}c\mathcal{D}\overline{c}\mathcal{D}B \exp\Big\{ i\int dx\Big[-i\delta_{\rm B}G(g^\dagger\partial_\mu g, c, \overline{c}, B) \\ &\quad + {\rm tr}\left(J_\mu\, g^\dagger\partial_\mu g + J_c c + J_{\overline{c}}\overline{c} + J_B B\right)\Big] + iW\left[gJ_\mu g^\dagger - \frac{4i}{\lambda}\delta_{\rm A}\delta_{\rm B}\left(g\partial_\mu g^\dagger\right)\right]\Big\}, \end{aligned} \tag{2.7}$$

where $W[j_\mu]$ is the generating functional of the connected Green's function of a_μ in the auxiliary field sector:

$$\begin{aligned} \exp iW[j_\mu] = \int \mathcal{D}a_\mu\mathcal{D}\gamma\mathcal{D}\overline{\gamma}\mathcal{D}\beta\mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\Big\{ i\int dx\Big[\frac{1}{2g_{\rm YM}^2}{\rm tr}F_{\mu\nu}^2(a) + \overline{\psi}\left(i\not{D}(a) - m\right)\psi \\ - i\widetilde{\delta}_{\rm B}H\left(a_\mu, \gamma, \overline{\gamma}, \beta\right) + {\rm tr}j_\mu a_\mu\Big]\Big\}. \end{aligned} \tag{2.8}$$

$W[j_\mu]$ should be calculated using ordinary perturbation theory with coupling constant $g_{\rm YM}^2$: the diagram contributing to $W[j_\mu]$ with N external a_μ and L auxiliary field loops is multiplied by $(g_{\rm YM}^2)^{N+L-1}$. Explicitly we have

$$W[j_\mu] = \frac{1}{2}g_{\rm YM}^2\, {\rm tr}\int dx\int dy j_\mu(x)D^{\mu\nu}(x-y)j_\nu(y) + O\left(g_{\rm YM}^4\right), \tag{2.9}$$

where $D_{\mu\nu}(x)$ is the free propagator in the Landau gauge:

$$D_{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - i\varepsilon}\Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)e^{ip\cdot x}. \tag{2.10}$$

In particular, from Eqs. (2.7) and (2.9) we reconfirm that the Yang-Mills theory is reduced to the PGM in the limit $g_{\rm YM}\to 0$. The case of the Green's functions containing the quark fields Ψ and $\overline{\Psi}$ can be treated similarly.

3 Color confinement

Now let us consider how the color confinement by the KO mechanism is realized in the perturbation expansion explained in the last section. Although the KO confinement condition is satisfied in PGM, there is no physical modes to be confined. Differently to the PGM the

present model does contain physical degrees of freedom, and we shall see that they are indeed confined owing to the disorder of the gauge-mode $g(x)$. Before starting the discussion, we here stress that we are interested in the confinement of the $\mathrm{SU}(N)_{\mathrm{R}}$ color charge carried by the original fields (A_μ, Ψ) and abandon the confinement of $\mathrm{SU}(N)_{\mathrm{L}}$ charge of (a_μ, ψ). These two color charges are physically equivalent and both should be confined in a more complete treatment of Yang-Mills theory. However, our perturbation expansion discriminates between these two color charges.

We shall discuss the confinement of $\mathrm{SU}(N)_{\mathrm{R}}$ color in two different ways: one is to see directly the absence of the physical asymptotic states of Ψ and A_μ, and the other is to study the sufficient condition for confinement, i.e., the absence of the Nambu-Goldstone mode coupled to the BRST-exact $\mathrm{SU}(N)_{\mathrm{R}}$ color current N_μ (1.3).

First, let us consider the asymptotic fields of quarks and gluons. The question is whether the two-point functions $\langle \mathrm{T}\Psi\overline{\Psi}\rangle$ and $\langle \mathrm{T}A_\mu A_\nu\rangle$ (in momentum space) have discrete poles corresponding to the physical asymptotic states. However, these discrete poles are absent in the lowest order of our perturbation expansion. This is seen as follows. Using the expression (2.3) for the original quark field Ψ in terms of ψ and g, we have

$$\left\langle \mathrm{T}\Psi_i(x)\overline{\Psi}_j(0)\right\rangle = \left\langle \mathrm{T}(g^\dagger)_{ik}(x) g_{lj}(0)\right\rangle_{\mathrm{PGM}} \left\langle \mathrm{T}\psi_k(x)\overline{\psi}_l(0)\right\rangle_0 + O(g_{\mathrm{YM}}^2), \tag{3.1}$$

where $\langle \mathrm{T}\cdots\rangle_{\mathrm{PGM}}$ denotes the Green's function in the PGM, and $\langle \mathrm{T}\psi_k\overline{\psi}_l\rangle_0 = \delta_{kl}/(\not{p} - m)$ is the free propagator ($i, j, \ldots$ are the $\mathrm{SU}(N)$ color indices in the fundamental representation). The point is that the PGM is in the disordered phase where $\mathrm{SU}(N)_{\mathrm{L}}\otimes\mathrm{SU}(N)_{\mathrm{R}}$ symmetry is realized linearly and the $\mathrm{SU}(N)$-valued field $g(x)$ has N^2 independent excitation modes; $\left\langle \mathrm{T}(g^\dagger)_{ik} g_{lj}\right\rangle_{\mathrm{PGM}} \sim \delta_{ij}\delta_{kl}/(p^2 - M^2)$ with M^2 being the mass gap of the PGM (in particular, we have $\langle g\rangle_{\mathrm{PGM}} = 0$). Namely, $\Psi = g^\dagger\psi$ is a genuine two-body composite operator and there is no interaction between the two in the lowest order in our expansion. Therefore, although the auxiliary quark field ψ has an asymptotic field, the original quark field Ψ does not. In a word, the large fluctuation of the gauge-mode g screens the asymptotic field of the auxiliary field ψ. The KO confinement mechanism requires only that the quark asymptotic fields, if they exist, are BRST unphysical [1]. In the present case, the asymptotic field of Ψ is totally absent.†

The absence of the pole in the Green's function $\left\langle \mathrm{T}\Psi\overline{\Psi}\right\rangle$ remains in general true in our perturbation theory so long as $\mathrm{SU}(N)_{\mathrm{L}}\otimes\mathrm{SU}(N)_{\mathrm{R}}$ is a good quantum number. The reason

† This implies that the asymptotic completeness [12], which requires that the Heisenberg fields are expressed in terms of the asymptotic fields, is lost if we restrict our consideration only to the original fields.

is as follows. A possible origin of the discrete pole in $\left\langle \mathrm{T}\Psi\overline{\Psi}\right\rangle$ is one-particle intermediate propagation of the auxiliary field ψ (see Fig. 1). However, this is impossible since ψ is $\mathrm{SU}(N)_{\mathrm{R}}$

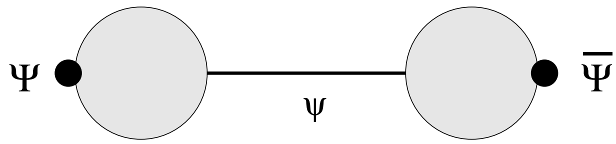

Figure 1: A possible origin of the discrete pole in $\langle \mathrm{T}\Psi\overline{\Psi}\rangle$. The solid line is the ψ propagator, and the shaded blobs represent complicated interactions.

singlet while Ψ is $\mathrm{SU}(N)_{\mathrm{R}}$ non-singlet. As for the gauge field A_μ, the situation is essentially the same as the quark field explained above. Note that, if the whole system including the PGM sector is treated in naive perturbation theory, the two-point function (3.1) has a discrete pole since we have $g \sim 1+\pi^a t^a$ with π^a ($a=1,\ldots,N^2-1$) being the Nambu-Goldstone mode.

Next, we shall look at the KO mechanism in the context of the absence of the Nambu-Goldstone mode coupled to the BRST-exact color current N_μ (1.3). This is almost trivially satisfied in any finite order in our perturbation expansion, since the $\mathrm{SU}(N)_{\mathrm{L}}\otimes\mathrm{SU}(N)_{\mathrm{R}}$ symmetry remains to be realized in a linear manner and N_μ is nothing but the Noether current of $\mathrm{SU}(N)_{\mathrm{R}}$. However, we shall see how it is satisfied more concretely. The following is almost the repetition of the argument for the quark confinement given above. The Nambu-Goldstone mode coupled to N_μ, if it exists, should be revealed as a massless pole in the two-point function $\langle \mathrm{T} N_\mu \mathcal{O}\rangle$ with $\mathcal{O}$ being an operator constructed from the original fields, for example $\mathcal{O}=A_\nu$ [1]. Within our perturbation theory the only possible origin of the massless pole contributing to this Green's function is the auxiliary gauge field a_μ with the free propagator (2.10). However, the diagrams having a single a_μ as its intermediate state vanish due to the conflict of the $\mathrm{SU}(N)_{\mathrm{L}}\otimes\mathrm{SU}(N)_{\mathrm{R}}$ quantum number (a_μ is $\mathrm{SU}(N)_{\mathrm{R}}$ singlet while N_μ and $\mathcal{O}$ are $\mathrm{SU}(N)_{\mathrm{R}}$ non-singlets).

Summarizing, what is important for (and almost equivalent to) the KO confinement mechanism is that the gauge-mode $g(x)$ is in the disordered phase and the $\mathrm{SU}(N)_{\mathrm{L}}\otimes\mathrm{SU}(N)_{\mathrm{R}}$ symmetry is realized in a linear manner. This property holds in the PGM and also in any finite order in our perturbation expansion. However, it is not clear at all whether this property persists beyond our perturbation expansion in g_{YM}, and this is our most important future subject.

4 Finite temperature system

In this section we shall apply our perturbation theory to the finite temperature case. In particular we are interested in what kind of PGM is obtained as the $g_{\rm YM} \to 0$ limit of finite temperature Yang-Mills theory. The partition function of the finite temperature Yang-Mills system is given by

$$Z_{\rm FT} = \int_{\rm periodic} \mathcal{D}A_\mu \mathcal{D}c\mathcal{D}\overline{c}\mathcal{D}B \exp\left\{ i\int dx \Big[\frac{1}{2g_{\rm YM}^2}{\rm tr}F_{\mu\nu}^2(A) - i\delta_{\rm B}G(A_\mu, c, \overline{c}, B) \Big] \right\}, \tag{4.1}$$

where all the path-integrals should be carried out using the periodic boundary condition for the complex time [13, 14]. We have omitted the quark fields from the system for simplicity. The following argument is applicable to both the imaginary and real time formalisms although we should consider the real time one if we are interested in the color (de)confinement at high temperature in the KO mechanism [15].

We shall generalize the Faddeev-Popov trick used in Sec. 2 to the finite temperature case. Eq. (2.2) for the FP determinant should now be modified to

$$1 = \Delta[A]\frac{1}{N}\sum_{k=0}^{N-1}\int_{B_k}\mathcal{D}g\prod_x \delta\left(\partial^\mu A_\mu^{g^{-1}}(x)\right), \tag{4.2}$$

where B_k denotes the boundary condition for $g(t, \boldsymbol{x})$ related by the Z_N twist:

$$B_k: \quad g(-i\beta, \boldsymbol{x}) = {\rm e}^{i\frac{2\pi k}{N}}g(0, \boldsymbol{x}) \quad (k = 0, 1, \cdots, N-1). \tag{4.3}$$

This boundary condition is understood from the fact that A_μ^g should obey the same periodic boundary condition as A_μ; $A_\mu^g(-i\beta, \boldsymbol{x}) = A_\mu^g(0, \boldsymbol{x})$. The reason why we have to sum over all the boundary conditions (4.3) is that there does not necessarily exist $g(t, \boldsymbol{x})$ which satisfies the gauge condition $\partial^\mu A_\mu^g = 0$ for an arbitrary A_μ if we restrict $g(t, \boldsymbol{x})$ to the periodic one. Since the summation in (4.2) is done with an equal weight, the FP determinant is invariant, $\Delta[A^h] = \Delta[A]$, under a gauge transformation h with the twisted boundary condition; $h(-i\beta, \boldsymbol{x}) = \exp(2\pi i l/N)h(0, \boldsymbol{x})$. Carrying out the same manipulation as in Sec. 2, we get the finite temperature generalization of Eq. (2.5) (we omit the source term):

$$\begin{aligned} Z_{\rm FT} = \frac{1}{N}\sum_{k=0}^{N-1}\int_{B_k}\mathcal{D}g\int_{\rm periodic}\mathcal{D}c\mathcal{D}\overline{c}\mathcal{D}B\int_{\rm periodic}\mathcal{D}a_\mu\mathcal{D}\gamma\mathcal{D}\overline{\gamma}\mathcal{D}\beta \\ \times \exp i\int dx\left[\frac{1}{2g_{\rm YM}^2}{\rm tr}F_{\mu\nu}^2(a) - i\tilde{\delta}_{\rm B}H(a_\mu, \gamma, \overline{\gamma}, \beta) - i\delta_{\rm B}G\left(g^\dagger(a_\mu + \partial_\mu)g, c, \overline{c}, B\right)\right]. \end{aligned} \tag{4.4}$$

Eq. (4.4) tells that the $g_{\rm YM} \to 0$ limit of the finite temperature Yang-Mills theory (without quark fields) is the PGM with summation over the Z_N boundary conditions:

$$Z_{\text{FT-PGM}} = \frac{1}{N}\sum_{k=0}^{N-1}\int_{B_k}\mathcal{D}g\int_{\text{periodic}}\mathcal{D}c\mathcal{D}\overline{c}\mathcal{D}B\exp\left\{i\int dx\left[-i\delta_{\rm B}G\left(g^\dagger\partial_\mu g, c, \overline{c}, B\right)\right]\right\}. \tag{4.5}$$

The same formula as (4.4) is also obtained using a different method. This is first to consider the partition function of a topological system of the set of fields $(g, \gamma, \overline{\gamma}, \beta)$:

$$Y[A_\mu] = \sum_{k=0}^{N-1}\int_{B_k}\mathcal{D}g\int_{\text{periodic}}\mathcal{D}\gamma\mathcal{D}\overline{\gamma}\mathcal{D}\beta\exp\left(-\widetilde{\delta}_{\rm B}H[A_\mu^{g^{-1}}, \gamma, \overline{\gamma}, \beta]\right). \tag{4.6}$$

This $Y[A_\mu]$ is in fact independent of A_μ. The invariance of (4.6) under an arbitrary infinitesimal deformation of A_μ, $A_\mu \to A_\mu + \delta A_\mu$, is a consequence of the $\widetilde{\delta}_{\rm B}$-exactness of the action and the $\widetilde{\delta}_{\rm B}$ invariance of the path-integral measure. Owing to the summation over all the boundary conditions (4.3) with an equal weight, $Y[A_\mu]$ is invariant under the "large" gauge transformation, $A_\mu \to A_\mu^h$ with $h(-i\beta, \boldsymbol{x}) = \exp(2\pi il/N)h(0, \boldsymbol{x})$. Since (4.6) is independent of A_μ, we insert (4.6) into Eq. (4.1), making the substitution (2.3), and we finally get Eq. (4.4).

The finite temperature PGM was discussed in Ref. [15] and the non-perturbative deconfining dynamics was analyzed there. Our finding was that in the low temperature region g is in the disordered (confining) phase, however, in the high temperature region the ordered (deconfining) phase is realized. The deconfining transition occurred since the infrared singularity in the perturbative phase is softened in the twisted boundary condition sectors. In Ref. [15] we summed over more general boundary conditions than (4.3); $g(-i\beta, \boldsymbol{x}) = h\cdot g(0, \boldsymbol{x})$ with h being a general element of the gauge group $\mathrm{SU}(N)$. However, the essential properties of the deconfining transition observed in Ref. [15] remain unchanged even if the boundary conditions are restricted to the Z_N ones as in (4.5).

If we introduce the quark field Ψ (which is subject to the anti-periodic boundary condition) into the system, the auxiliary quark field $\psi = g\Psi$ in the B_k sector should obey the boundary condition $\psi(-i\beta, \boldsymbol{x}) = -\exp(2\pi ik/N)\,\psi(0, \boldsymbol{x})$. The (de)confinement of the original quark field Ψ is shown similarly to the zero-temperature case of Sec. 3.

5 Discussion

In this paper we have presented a perturbation expansion to incorporate the physical degrees of freedom into the PGM. We have seen that the quarks and gluons are confined in this perturbation theory.

Of course this is not the end of the confinement problem. We have to know whether the confining property persists in a more complete analysis of QCD including the non-perturbative treatment in $g_{\rm YM}$. In particular, we are interested in whether the disordered phase of the gauge-mode $g(x)$ remains intact beyond perturbation theory.[‡] For studying non-perturbative effects to the dynamics of the gauge-mode, it may be helpful to consider the "effective action" $S_{\rm eff}$ by regarding (2.7) with $J=0$ as a modified topological model:

$$S_{\rm eff}(g,c,\overline{c},B) = -i\int dx \delta_{\rm B} G(g^\dagger\partial_\mu g, c, \overline{c}, B) + W\left[-\frac{4i}{\lambda}\delta_{\rm A}\delta_{\rm B}\left(g\partial_\mu g^\dagger\right)\right]. \tag{5.1}$$

Note that the W term in (5.1) is also written in a BRST-exact form $\delta_{\rm B}(\cdots)$.

As a related question, we have not yet understood how the confinement breaks down in the Higgs phase. Namely, if we introduce the Higgs scalar into the system and break the color symmetry spontaneously, it is natural to expect that the color confinement fails. However, in the lowest order in our perturbation theory, the color confinement holds irrespectively of the presence of the Higgs field. This problem may be understood on the basis of the effective topological model (5.1). In the case the color gauge symmetry is broken in an asymmetric manner and the gauge bosons get different masses, this effect appears in W of (5.1) as explicit breaking terms of the $\mathrm{SU}(N)_{\rm L}$ symmetry as seen from the lowest order expression (2.9). This explicit breaking may trigger the ordered phase of the gauge-mode $g(x)$ to break the confinement. However, if the gauge bosons acquire an equal mass (for example, the case the SU(2) gauge symmetry is broken by a doublet Higgs field), the W term of (5.1) has no explicit breaking of $\mathrm{SU}(N)_{\rm L}\otimes\mathrm{SU}(N)_{\rm R}$. The understanding of the failure of confinement in this case is a challenging problem in the future work.

Appendix: BRST transformations

Here, we summarize the (anti-)BRST transformations, $\delta_{\rm B}$, $\delta_{\rm A}$, $\widetilde{\delta}_{\rm B}$ and $\widetilde{\delta}_{\rm A}$. First, $\delta_{\rm B}$ and $\delta_{\rm A}$ are defined by

$$\delta_{\rm B}A_\mu = D_\mu(A)c,\quad \delta_{\rm B}g = gc,\quad \delta_{\rm B}c = -c^2,\quad \delta_{\rm B}\overline{c} = iB,\quad \delta_{\rm B}B = 0,\quad \delta_{\rm B}\Psi = c\Psi, \tag{A.1}$$

$$\delta_{\rm A}A_\mu = -D_\mu(A)\overline{c},\quad \delta_{\rm A}g = -g\overline{c},\quad \delta_{\rm A}\overline{c} = \overline{c}^2,\quad \delta_{\rm A}c = -i\overline{B},\quad \delta_{\rm A}\overline{B} = 0,\quad \delta_{\rm A}\Psi = -\overline{c}\Psi, \tag{A.2}$$

‡ As good news for confinement, we mention the one-loop beta function of the PGM coupling constant λ, $\beta_\lambda = d\lambda/d\ln\mu = -\lambda^2 - 3g_{\rm YM}^2\lambda$ [16]. This β_λ tells that $g_{\rm YM}$ helps to strengthen the infrared slavery of λ.

with $B+\overline{B}=i\{c,\overline{c}\}$. The auxiliary fields $(a_\mu,\gamma,\overline{\gamma},\beta,\psi)$ are inert under $\delta_{\rm B}$ and $\delta_{\rm A}$. Next are $\widetilde{\delta}_{\rm B}$ and $\widetilde{\delta}_{\rm A}$:

$$\widetilde{\delta}_{\rm B}a_\mu=D_\mu(a)\gamma,\quad \widetilde{\delta}_{\rm B}g=-\gamma g,\quad \widetilde{\delta}_{\rm B}\gamma=-\gamma^2,\quad \widetilde{\delta}_{\rm B}\overline{\gamma}=i\beta,\quad \widetilde{\delta}_{\rm B}\beta=0,\quad \widetilde{\delta}_{\rm B}\psi=\gamma\psi, \tag{A.3}$$
$$\widetilde{\delta}_{\rm A}a_\mu=-D_\mu(a)\overline{\gamma},\quad \widetilde{\delta}_{\rm A}g=\overline{\gamma}g,\quad \widetilde{\delta}_{\rm A}\overline{\gamma}=\overline{\gamma}^2,\quad \widetilde{\delta}_{\rm A}\overline{\gamma}=-i\overline{\beta},\quad \widetilde{\delta}_{\rm A}\overline{\beta}=0,\quad \widetilde{\delta}_{\rm A}\psi=-\overline{\gamma}\psi, \tag{A.4}$$

where $\beta+\overline{\beta}=i\{\gamma,\overline{\gamma}\}$. The original fields $(A_\mu,c,\overline{c},B,\Psi)$ are invariant under $\widetilde{\delta}_{\rm B}$ and $\widetilde{\delta}_{\rm A}$.

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